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# Acoustic diffraction by two concentric coaxial soft spherical caps

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SUMMARY
The subject of this paper is the problem of diffraction of a time-harmonic axially symmetric acoustic wave by two concentric coaxial soft spherical caps. An integral equation technique is employed to solve such a boundary value problem involving two concentric coaxial spherical caps. Approximate expressions are derived for the far field amplitude as well as the scattering cross section for this problem when the incident wave is a low frequency axially symmetric plane wave travelling along the common axis of the two caps. By taking appropriate limits, the formulae for scattering cross section for the corresponding problems for a soft spherical cap, a soft sphere and a soft sphere bounded by a concentric soft spherical cap are also derived. Furthermore, the total electrostatic charge required to raise the two concentric coaxial spherical caps to unit potentials in a free space is readily evaluated from the analysis of this paper.

## 1. Introduction

Recently some research workers have discussed the problems of acoustic diffraction by a soft or a rigid spherical cap [1, 2, 3]. Jain and Kanwal [4, 5] have also solved the problems of acoustic diffraction by a soft or a rigid annular spherical cap. Although Collins [6] and Vaid and Jain [7] have presented the solutions of the problems of acoustic diffraction by two coaxial soft and rigid circular disks, yet no attempt has been made so far to solve the problems of acoustic diffraction by two concentric coaxial soft or rigid spherical caps. We discuss here the solution of the problem of diffraction of a time-harmonic axially symmetric acoustic wave by two concentric coaxial soft spherical caps.

The method of solution rests on formulating the problem in terms of two simultaneous Fredholm integral equations of the first kind which embody the steady state wave equation as well as the boundary conditions. An integral equation technique [8] is employed to reduce the two governing simultaneous Fredholm integral equations of the first kind to four Volterra integral equations of the first kind and two simultaneous Fredholm integral equations of the second kind. The four Volterra integral equations have a simple kernel and therefore can be easily inverted, while the two simultaneous Fredholm integral equations of the second kind can be readily solved by the standard iterative procedure when the frequency of the incident wave is low and the radius of the inner cap is small as compared to that of the outer cap. The formula for the far field amplitude is also derived.

The formulation as well as the solution is given for an axially symmetric acoustic wave. A detailed discussion is then presented for the special case when the incident wave is an axially symmetric plane wave travelling along the common axis of the two caps. Approximate expressions for the far field amplitude as well as the scattering cross section are derived for this particular case. By taking some appropriate limits, we readily obtain the formulae for scattering cross section of a soft spherical cap as well as a soft sphere and these formulae agree with the known results [1–4]. But the formula for scattering cross section for a soft sphere bounded by a concentric soft spherical cap obtained by taking an appropriate limit seems to be new. Finally, the analysis of this paper is used to evaluate the total electrostatic charge required to raise the two coaxial concentric caps to unit potentials in a free space.

## 2. Formulation of the problem

We take the common centre and the common axis of the two concentric coaxial soft spherical caps of semi-angles $\alpha_1$, $\alpha_2$ as the origin and polar axis for spherical polar coordinates $(r, \theta, \varphi)$ so that the two caps are defined by $r=a_1$, $0 \leqq \theta \leqq \alpha_1$, all $\varphi$ and $r=a_2$, $0 \leqq \theta \leqq \alpha_2$, all $\varphi$, where $a_j$, $(j=1, 2)$ are the radii of the two caps. It is assumed that an axially symmetric acoustic wave whose velocity potential is $u_0(r, \theta)\mathrm{e}^{-i\omega t}$ impinges on the caps. The total velocity potential is of the form $\{u_0(r, \theta)+\phi(r, \theta)\}\,\mathrm{e}^{-i\omega t}$. Both $u_0$ and $\phi$ satisfy the Helmholtz equation

$$(\nabla^2+k^2)u(r, \theta) = 0\,, \quad k^2 = \omega^2/c^2\,, \tag{1}$$

where $c$ is the speed of wave propagation. We have to solve the following boundary value problem:

$$(\nabla^2+k^2)\phi(r, \theta) = 0\,, \tag{2}$$

$$\phi(a_1, \theta) = -u_0(a_1, \theta)\,, \quad 0 \leqq \theta \leqq \alpha_1\,, \tag{3}$$

$$\phi(a_2, \theta) = -u_0(a_2, \theta)\,, \quad 0 \leqq \theta \leqq \alpha_2\,, \tag{4}$$

$\phi$, $\partial\phi/\partial r$ are continuous across the regions $r=a_1$, $\alpha_1 < \theta \leqq \pi$, all $\varphi$ and

$$r = a_2\,, \ \alpha_2 < \theta \leqq \pi\,, \ \text{all } \varphi\,. \tag{5}$$

In addition, $\phi(r, \theta)$ satisfies the Sommerfeld radiation condition.

The integral representation formula for $\phi$ which embodies the steady state wave equation (2) and continuity requirements (5) is

$$\phi(r, \theta) = -\frac{a_1^2}{4\pi}\int_0^{\alpha_1}\int_0^{2\pi} g_1(t)\left[\frac{\mathrm{e}^{ikR}}{R}\right]_{r_1=a_1} d\varphi_1\, dt - \frac{a_2^2}{4\pi}\int_0^{\alpha_2}\int_0^{2\pi} g_2(t)\left[\frac{\mathrm{e}^{ikR}}{R}\right]_{r_1=a_2} d\varphi_1 dt\,, \tag{6}$$

where

$$g_{1,2}(t) = \sin t\left\{\left[\frac{\partial\phi(r_1, t)}{\partial r_1}\right]_{r_1=a_1+,\,a_2+} - \left[\frac{\partial\phi(r_1, t)}{\partial r_1}\right]_{r_1=a_1-,\,a_2-}\right\}, \tag{7}$$

$R$ is the distance between the source point $(r_1, t, \varphi_1)$ and the field point $(r, \theta, \varphi)$, that is, $R=(r^2+r_1^2-2rr_1\cos\psi)^{\frac{1}{2}}$ and $\cos\psi=\cos\theta\cos t+\sin\theta\sin t\cos(\varphi-\varphi_1)$.

When we use the boundary conditions (3) and (4) in (6), we obtain two simultaneous Fredholm integral equations of the first kind for evaluating the unknown density functions $g_j$, $(j=1, 2)$

$$\int_0^{\alpha_1} K_{11}(t, \theta)g_1(t)dt + \int_0^{\alpha_2} G_{12}(t, \theta)g_2(t)dt = u_0(a_1, \theta) = f_1(\theta)\,, \quad 0 \leqq \theta \leqq \alpha_1\,, \tag{8}$$

$$\int_0^{\alpha_1} G_{21}(t, \theta)g_1(t)dt + \int_0^{\alpha_2} K_{22}(t, \theta)g_2(t)dt = u_0(a_2, \theta) = f_2(\theta)\,, \quad 0 \leqq \theta \leqq \alpha_2\,, \tag{9}$$

where

$$K_{jj}(t, \theta) = \frac{a_j^2}{4\pi}\int_0^{2\pi}\left[\frac{\mathrm{e}^{ikR}}{R}\right]_{r_1=a_j,\, r=a_j} d\varphi_1\,, \quad (j = 1, 2)\,, \tag{10}$$

$$G_{jl}(t, \theta) = \frac{a_l^2}{4\pi}\int_0^{2\pi}\left[\frac{\mathrm{e}^{ikR}}{R}\right]_{r_1=a_l,\, r=a_j} d\varphi_1\,, \quad (j \neq l;\, j, l = 1, 2)\,. \tag{11}$$

The two governing simultaneous Fredholm integral equations of the first kind for this problem (8) and (9) are similar to equations (2.1) for $n=2$ in [8]. So in order to apply the integral equation technique [8], we use in (10) and (11) the well known formulae [9, 10]

$$\frac{\mathrm{e}^{ikR}}{R} = ik\sum_{n=0}^{\infty}(2n+1)j_n(kr\langle)h_n(kr\rangle)P_n(\cos\psi)\,, \tag{12}$$

$$\int_0^{2\pi} P_n(\cos\psi)\,d\varphi_1 = 2\pi P_n(\cos\theta)P_n(\cos t)\,, \tag{13}$$

$$P_n(\cos\theta) = \frac{(2)^{\frac{1}{2}}}{\pi}\int_0^{\theta}\frac{\cos\{(n+\frac{1}{2})w\}\,dw}{(\cos w-\cos\theta)^{\frac{1}{2}}}\,, \tag{14}$$

$$\frac{2}{\pi}\sum_{n=0}^{\infty}\cos\{(n+\tfrac{1}{2})w\}\cos\{(n+\tfrac{1}{2})v\}=\delta(w-v)\,,\quad 0<v\,,\ w<\pi\,, \tag{15}$$

where

$$r\langle=\min(r,r_1)\,,\quad r\rangle=\max(r,r_1)\,,$$

$$j_n(x)=(\pi/2x)^{\frac{1}{2}}J_{n+\frac{1}{2}}(x)\,,\quad h_n(x)=(\pi/2x)^{\frac{1}{2}}H^{(1)}_{n+\frac{1}{2}}(x)\,,$$

$J$, $H^{(1)}$, $P$ and $\delta$ are the Bessel function, the Hankel function of the first kind, the Legendre polynomial and the Dirac delta function respectively. There results after some simplifications

$$K_{jj}(t,\theta)=K_j(t,\theta)+G_{jj}(t,\theta)\,,\quad (j=1,2)\,, \tag{16}$$

$$K_j(t,\theta)=\tfrac{1}{2}a_j\sum_{n=0}^{\infty}P_n(\cos\theta)P_n(\cos t)=\frac{1}{2\pi}a_j\int_0^{\min(\theta,t)}\frac{dw}{(\cos w-\cos\theta)^{\frac{1}{2}}(\cos w-\cos t)^{\frac{1}{2}}}\,,\quad (j=1,2)\,, \tag{17}$$

$$G_{jl}(t,\theta)=\frac{1}{2\pi}a_l\int_0^{\theta}\int_0^{t}\frac{L_{jl}(v,w)\,dv\,dw}{(\cos w-\cos\theta)^{\frac{1}{2}}(\cos v-\cos t)^{\frac{1}{2}}}\,,\quad (j,l=1,2)\,, \tag{18}$$

where

$$L_{jj}(v,w)=\frac{2}{\pi}\sum_{n=0}^{\infty}[ika_j(2n+1)j_n(ka_j)h_n(ka_j)-1]\cos\{(n+\tfrac{1}{2})w\}\cos\{(n+\tfrac{1}{2})v\}\,,\quad (j=1,2)\,, \tag{19}$$

$$L_{12}(v,w)=\frac{2ika_2}{\pi}\sum_{n=0}^{\infty}(2n+1)j_n(ka_1)h_n(ka_2)\cos\{(n+\tfrac{1}{2})w\}\cos\{(n+\tfrac{1}{2})v\}\,, \tag{20}$$

$$L_{21}(v,w)=(a_1/a_2)L_{12}(v,w)\,,\ a_1<a_2\,. \tag{21}$$

It follows from relations (16) to (18) that the kernels $K_{jj}$ and $G_{jl}$, $(j,l=1,2)$ satisfy all the requirements for the application of the integral equation technique [8] and we have in this case $\gamma_j=a_j$, $(j=1,2)$, $h_1(\theta)=(2\pi)^{-1}$, $h_2(\theta)=h_3(\theta)=1$, $K(w,\theta)=(\cos w-\cos\theta)^{-\frac{1}{2}}$, and the kernels $L_{jl}$, $(j,l=1,2)$ are given by relations (19) to (21). The kernel $K(w,\theta)=(\cos w-\cos\theta)^{-\frac{1}{2}}$ is an elementary function and therefore the Volterra integral equations of the first kind (2.4) and (2.5) in [8] can be easily inverted by using the well known formulae [9, 11]. Hence by the integral equation technique [8], the two simultaneous Fredholm integral equations of the first kind (8) and (9) can be reduced to the following four Volterra integral equations of the first kind and two Fredholm integral equations of the second kind:

$$S_j(\theta)=a_j\int_{\theta}^{\alpha_j}\frac{g_j(t)\,dt}{(\cos\theta-\cos t)^{\frac{1}{2}}}\,,\quad 0\leqq\theta\leqq\alpha_j\,,\quad (j=1,2)\,, \tag{22}$$

$$\frac{1}{2\pi}\int_0^{\theta}\frac{C_j(t)\,dt}{(\cos t-\cos\theta)^{\frac{1}{2}}}=f_j(\theta)\,,\quad 0\leqq\theta\leqq\alpha_j\,,\quad (j=1,2)\,, \tag{23}$$

$$S_1(\theta)+\int_0^{\alpha_1}L_{11}(v,\theta)S_1(v)dv+\int_0^{\alpha_2}L_{12}(v,\theta)S_2(v)dv=C_1(\theta)\,,\quad 0\leqq\theta\leqq\alpha_1\,, \tag{24}$$

$$S_2(\theta)+\int_0^{\alpha_1}L_{21}(v,\theta)S_1(v)dv+\int_0^{\alpha_2}L_{22}(v,\theta)S_2(v)dv=C_2(\theta)\,,\quad 0\leqq\theta\leqq\alpha_2\,, \tag{25}$$

where the kernels $L_{jl}$, $(j,l=1,2)$ are given by equations (19) to (21). The Volterra integral equations of the first kind (22) and (23) are first inverted to yield [9, 11]

$$a_jg_j(t)=-\pi^{-1}\frac{d}{dt}\int_t^{\alpha_j}\frac{\sin u\,S_j(u)\,du}{(\cos t-\cos u)^{\frac{1}{2}}}\,,\quad (j=1,2)\,, \tag{26}$$

$$C_j(\theta)=2\frac{d}{d\theta}\int_0^{\theta}\frac{\sin u\,f_j(u)\,du}{(\cos u-\cos\theta)^{\frac{1}{2}}}\,,\quad (j=1,2)\,. \tag{27}$$

After substituting the values of $C_j$, $(j=1,2)$ in terms of the known functions $f_j$, $(j=1,2)$ from (27) in (24) and (25), the resulting equations can be solved simultaneously to determine the values of the functions $S_j$, $(j=1,2)$ in terms of the small dimensionless parameters $\varepsilon=a_1/a_2$ and $\gamma=ka_2$, by the method of successive approximations. Consequently equations (26) lead to the required values of the unknown density functions $g_j$, $(j=1,2)$.

## 3. Far field amplitude

The far field amplitude $A(\theta)$ is defined as

$$\phi(r,\theta)=A(\theta)\frac{e^{ikr}}{r}+O(r^{-2}). \tag{28}$$

Comparing it with the integral representation formula (6), we obtain

$$A(\theta)=-\frac{a_1^2}{4\pi}\int_0^{\alpha_1}\int_0^{2\pi}g_1(t)e^{-i\gamma\varepsilon\cos\psi}d\varphi_1\,dt-\frac{a_2^2}{4\pi}\int_0^{\alpha_2}\int_0^{2\pi}g_2(t)e^{-i\gamma\cos\psi}d\varphi_1\,dt. \tag{29}$$

When we use the relation [10]

$$e^{-i\gamma\cos\psi}=\sum_{n=0}^{\infty}(-i)^n(2n+1)j_n(\gamma)P_n(\cos\psi), \tag{30}$$

formulae (13) and (26) in (29), there results

$$\begin{aligned}A(\theta)=\;&\frac{-a_1}{(2)^{\frac{1}{2}}\pi}\sum_{n=0}^{\infty}(-i)^n(2n+1)j_n(\gamma\varepsilon)P_n(\cos\theta)\int_0^{\alpha_1}S_1(w)\cos\{(n+\tfrac{1}{2})w\}\,dw\\&-\frac{a_2}{(2)^{\frac{1}{2}}\pi}\sum_{n=0}^{\infty}(-i)^n(2n+1)j_n(\gamma)P_n(\cos\theta)\int_0^{\alpha_2}S_2(w)\cos\{(n+\tfrac{1}{2})w\}\,dw.\end{aligned} \tag{31}$$

Note that in view of the above results, Sommerfeld's radiation condition is satisfied by $\phi$.

## 4. Special case

We present here the solution of the problem for the special case when the incident wave is a low-frequency plane wave travelling along the negative direction of the polar axis. It is further assumed that the radius of inner cap is small as compared with that of the outer cap. So the small dimensionless perturbation parameters in this analysis are $\varepsilon=a_1/a_2$ and $\gamma=ka_2$. It is further assumed that $\gamma=O(\varepsilon)$ and therefore $ka_1=\gamma\varepsilon=O(\gamma^2)$. We have in this special case

$$u_0(r,\theta)=e^{-ikr\cos\theta},\quad f_1(\theta)=e^{-i\gamma\varepsilon\cos\theta},\quad f_2(\theta)=e^{-i\gamma\cos\theta}.$$

Then it follows from relations (27) that

$$\begin{aligned}C_1(\theta)&=(2)^{\frac{3}{2}}\sum_{n=0}^{\infty}(-i)^n(2n+1)j_n(\gamma\varepsilon)\cos\{(n+\tfrac{1}{2})\theta\}\\&=(2)^{\frac{3}{2}}\left[\cos\frac{\theta}{2}-i\gamma\varepsilon\cos\frac{3\theta}{2}+O(\gamma^4)\right],\end{aligned} \tag{32}$$

$$\begin{aligned}C_2(\theta)&=(2)^{\frac{3}{2}}\sum_{n=0}^{\infty}(-i)^n(2n+1)j_n(\gamma)\cos\{(n+\tfrac{1}{2})\theta\}\\&=(2)^{\frac{3}{2}}\left[\left(1-\frac{\gamma^2}{6}\right)\cos\frac{\theta}{2}-i\gamma\cos\frac{3\theta}{2}-\frac{\gamma^2}{3}\cos\frac{5\theta}{2}+O(\gamma^3)\right],\end{aligned} \tag{33}$$

where we have used the expansion (30) as well as the formula [10]

$$\frac{d}{d\theta}\int_0^{\theta}\frac{\sin u\,P_n(\cos u)\,du}{(\cos u-\cos\theta)^{\frac{1}{2}}}=(2)^{\frac{1}{2}}\cos\{(n+\tfrac{1}{2})\theta\}. \tag{34}$$

Similarly the kernels $L_{jl}$, $(j, l=1, 2)$ can be easily expanded in terms of the small parameters $\varepsilon$ and $\gamma$ and we have from relations (19) to (21)

$$L_{11}(v, w) = \frac{2}{\pi}\left[ i\gamma\varepsilon \cos\frac{v}{2}\cos\frac{w}{2} + O(\gamma^3)\right], \tag{35}$$

$$L_{22}(v, w) = \frac{2i\gamma}{\pi}\cos\frac{v}{2}\cos\frac{w}{2} - \begin{cases}\frac{1}{2}\gamma^2 \sin w \cos v\,, & v<w\\ \frac{1}{2}\gamma^2 \sin v \cos w\,, & v>w\end{cases} + O(\gamma^3), \tag{36}$$

$$L_{12}(v, w) = \frac{2}{\pi}\left[\left(1+i\gamma - \frac{\gamma^2}{2}\right)\cos\frac{v}{2}\cos\frac{w}{2} + \varepsilon\cos\frac{3v}{2}\cos\frac{3w}{2} + \right.$$
$$\left. + \varepsilon^2\cos\frac{5v}{2}\cos\frac{5w}{2} + O(\gamma^3)\right], \tag{37}$$

$$L_{21}(v, w) = \varepsilon L_{12}(v, w)$$
$$= \frac{2}{\pi}\left[(\varepsilon+i\gamma\varepsilon)\cos\frac{v}{2}\cos\frac{w}{2} + \varepsilon^2\cos\frac{3v}{2}\cos\frac{3w}{2} + O(\gamma^3)\right]. \tag{38}$$

After substituting the above expansions of the functions $C_j(\theta)$, $(j=1, 2)$ and the kernels $L_{jl}$, $(j, l=1, 2)$ in equations (24) and (25), we obtain after solving these equations simultaneously by the method of successive approximations

$$S_j(\theta) = (2)^{\frac{3}{2}}\left[\left\{ a_j'\cos\frac{\theta}{2} + \left(b_j'\cos\frac{\theta}{2} + b_j''\cos\frac{3\theta}{2}\right)\varepsilon + \left(c_j'\cos\frac{\theta}{2} + c_j''\cos\frac{3\theta}{2} + \right.\right.\right.$$
$$\left. + c_j'''\cos\frac{5\theta}{2}\right)\varepsilon^2 + \left(d_j'\cos\frac{\theta}{2} + d_j''\cos\theta + d_j'''\cos\frac{5\theta}{2}\right)\gamma^2 + O(\gamma^3)\Bigg\} +$$
$$\left. + i\left\{\left(e_j'\cos\frac{\theta}{2} + e_j''\cos\frac{3\theta}{2}\right)\gamma + \left(f_j'\cos\frac{\theta}{2} + f_j''\cos\frac{3\theta}{2}\right)\gamma\varepsilon + O(\gamma^3)\right\}\right],$$
$$(j=1, 2), \tag{39}$$

where

$$a_1' = 1 - W_0(\alpha_2)\,,$$
$$b_1' = W_0(\alpha_1)\, W_0(\alpha_2)\{1 - W_0(\alpha_2)\}\,, \quad b_1'' = -W_1(\alpha_2)\,,$$
$$c_1' = W_1(\alpha_1)\, W_1(\alpha_2)\{1 - 2\,W_0(\alpha_2)\} + W_0^2(\alpha_1)\, W_0^2(\alpha_2)\{1 - W_0(\alpha_2)\}\,,$$
$$c_1'' = W_0(\alpha_1)\, W_1(\alpha_2)\{1 - W_0(\alpha_2)\}\,, \quad c_1''' = -W_2(\alpha_2)\,,$$
$$d_1' = -W_1(\alpha_2) + \tfrac{1}{3}W_2(\alpha_2) + W_0(\alpha_2)\{W_1(\alpha_2) - W_0(\alpha_2) + W_0^2(\alpha_2)\} +$$
$$+ \frac{1}{\pi}\left(\frac{1}{3}\cos\frac{3\alpha_2}{2} + \cos\frac{\alpha_2}{2}\right)\left(\frac{1}{3}\sin\frac{3\alpha_2}{2} + \sin\frac{\alpha_2}{2}\right), \quad d_1'' = 0\,, \quad d_1''' = 0\,,$$
$$e_1' = W_0(\alpha_2)\{-1 + W_0(\alpha_2)\} + W_1(\alpha_2)\,, \quad e_1'' = 0\,,$$
$$f_1' = W_0(\alpha_1)\{-1 + 3W_0(\alpha_2) - 4\,W_0^2(\alpha_2) + W_0(\alpha_2)\, W_1(\alpha_2) + 2\,W_0^3(\alpha_2)\}\,,$$
$$f_1'' = -1 + \frac{1}{\pi}\left(\alpha_2 + \frac{\sin 3\alpha_2}{3}\right) + W_0(\alpha_2)\, W_1(\alpha_2)\,;$$

$$a_2' = 1\,,$$
$$b_2' = W_0(\alpha_1)\{-1 + W_0(\alpha_2)\}\,, \quad b_2'' = 0\,,$$
$$c_2' = W_1(\alpha_1)\, W_1(\alpha_2) - W_0(\alpha_2)\, W_0^2(\alpha_1)\{1 - W_0(\alpha_2)\}\,,$$
$$c_2'' = W_1(\alpha_1)\{-1 + W_0(\alpha_2)\}\,, \quad c_2''' = 0\,,$$

$$d_2' = \tfrac{1}{2} - W_1(\alpha_2) - W_0^2(\alpha_2)\,, \quad d_2'' = -\tfrac{1}{2}\left(\tfrac{1}{3}\cos\frac{3\alpha_2}{2} + \cos\frac{\alpha_2}{2}\right), \quad d_2''' = -\tfrac{1}{3}\,,$$

$$e_2' = -W_0(\alpha_2)\,, \quad e_2'' = -1\,,$$

$$f_2' = W_0(\alpha_1)\{-1 - W_1(\alpha_2) + 3W_0(\alpha_2) - 2W_0^2(\alpha_2)\}\,, \quad f_2'' = 0\,,$$

and

$$W_n(\theta) = \begin{cases} \dfrac{1}{\pi}\left[\dfrac{\sin n\theta}{n} + \dfrac{\sin(n+1)\theta}{n+1}\right], & n \geqq 1 \\ \dfrac{1}{\pi}\,[\theta + \sin\theta]\,, & n = 0\,. \end{cases}$$

We may point out here that although the kernel $L_{12}(v, w)$ defined by (37) does not tend to zero when the perturbation parameters $\gamma, \varepsilon$ tend to zero, yet the method of successive approximations can be used to obtain an approximate solution (39) of the simultaneous Fredholm integral equations (24) and (25).

When we substitute the above values of $S_j$, $(j=1, 2)$ in the formula for the far field amplitude (31), we obtain

$$\begin{aligned} A(\theta) = -a_2 \Bigg[\Bigg\{ & W_0(\alpha_2)a_2' + (W_0(\alpha_2)b_2' + W_0(\alpha_1)a_1')\varepsilon + (W_0(\alpha_1)b_1' + W_1(\alpha_1)b_1'' + \\ & + W_0(\alpha_2)c_2' + W_1(\alpha_2)c_2'')\varepsilon^2 + \Bigg(\Bigg(W_0(\alpha_2)d_2' + \frac{1}{\pi}\left(\tfrac{2}{3}\sin\frac{3\alpha_2}{2} + 2\sin\frac{\alpha_2}{2}\right)d_2'' + \\ & + W_2(\alpha_2)d_2''' - \tfrac{1}{6}W_0(\alpha_2)a_2'\Bigg) + \left(W_1(\alpha_2)e_2' + \frac{1}{\pi}\left(\alpha_2 + \frac{\sin 3\alpha_2}{3}\right)e_2''\right)\cos\theta - \\ & - \tfrac{1}{6}W_2(\alpha_2)a_2'(3\cos^2\theta - 1)\Bigg)\gamma^2 + O(\gamma^3)\Bigg\} + \mathrm{i}\{(W_0(\alpha_2)e_2' + W_1(\alpha_2)e_2'' - \\ & - W_1(\alpha_2)a_2'\cos\theta)\gamma + (W_0(\alpha_1)e_1' + W_0(\alpha_2)f_2' - W_1(\alpha_2)b_2'\cos\theta)\gamma\varepsilon + O(\gamma^3)\}\Bigg]\,, \end{aligned} \tag{40}$$

where $a_j'$, $b_j'$, $b_j''$, $c_j'$, $c_j''$, $c_j'''$, $d_j'$, $d_j''$, $d_j'''$, $e_j'$, $e_j''$, $f_j'$, $f_j''$, $(j=1, 2)$, are given by (39). Finally, we put this value of $A(\theta)$ in formula for the scattering cross section.

$$\sigma = 2\pi \int_0^\pi |A(\theta)|^2 \sin\theta\, d\theta\,, \tag{41}$$

and we obtain after some simplifications

$$\begin{aligned} \sigma = \frac{4a_2^2}{\pi}\Bigg[ & (\alpha_2 + \sin\alpha_2)^2 + 2(\alpha_1 + \sin\alpha_1)(\alpha_2 + \sin\alpha_2)\left\{1 - \frac{1}{\pi}(\alpha_2 + \sin\alpha_2)\right\}^2 \varepsilon + \\ & + \Bigg((\alpha_1 + \sin\alpha_1)^2\left\{1 - \frac{4}{\pi}(\alpha_2 + \sin\alpha_2) + \frac{3}{\pi^4}(\alpha_2 + \sin\alpha_2)^4\right\} + \frac{4}{\pi}(\alpha_2 + \sin\alpha_2)\times \\ & \times\left\{1 - \frac{1}{\pi}(\alpha_2 + \sin\alpha_2)\right\}\Bigg\{\frac{2}{\pi}(\alpha_1 + \sin\alpha_1)^2(\alpha_2 + \sin\alpha_2) - \\ & - \left(\sin\alpha_1 + \frac{\sin 2\alpha_1}{2}\right)\left(\sin\alpha_2 + \frac{\sin 2\alpha_2}{2}\right)\Bigg\}\Bigg)\varepsilon^2 + \\ & + \tfrac{1}{3}\Big\{(2\sin\alpha_2 + \sin 2\alpha_2)^2 - (\alpha_2 + \sin\alpha_2)(-2\alpha_2 + \sin\alpha_2 + 3\sin 2\alpha_2 + \sin 3\alpha_2) - \\ & - \frac{3}{\pi^2}(\alpha_2 + \sin\alpha_2)^4 \quad \gamma^2 + O(\gamma^3)\Bigg]\,. \end{aligned} \tag{42}$$

When we let (i) $a_1 \to 0$ or $\alpha_1 \to 0$; (ii) $\alpha_2 \to 0$; (iii) $\alpha_1 \to 0, \alpha_2 \to \pi$ or $\alpha_1 \to \pi, \alpha_2 \to 0$ in the above result, we readily obtain the limiting formulae for the scattering cross section of a soft spherical cap or a soft sphere which agree with the known results [1–4]. But when we let $\alpha_1 \to \pi$ in (42), we derive the formula for the scattering cross section $\sigma'$ of the corresponding problem for a soft sphere bounded by a concentric soft spherical cap

$$\sigma' = \frac{4a_2^2}{\pi}\Bigg[(\alpha_2+\sin\alpha_2)^2+2\pi(\alpha_2+\sin\alpha_2)\left\{1-\frac{1}{\pi}(\alpha_2+\sin\alpha_2)\right\}^2\varepsilon+$$
$$+\left(\pi^2\left\{1-\frac{4}{\pi}(\alpha_2+\sin\alpha_2)+\frac{3}{\pi^4}(\alpha_2+\sin\alpha_2)^4\right\}+\right.$$
$$\left.+\,8(\alpha_2+\sin\alpha_2)^2\left\{1-\frac{1}{\pi}(\alpha_2+\sin\alpha_2)\right\}\right)\varepsilon^2+\tfrac{1}{3}\Big\{(2\sin\alpha_2+\sin 2\alpha_2)^2-$$
$$-(\alpha_2+\sin\alpha_2)(-2\alpha_2+\sin\alpha_2+3\sin 2\alpha_2+\sin 3\alpha_2)-\frac{3}{\pi^2}(\alpha_2+\sin\alpha_2)^4\Big\}\gamma^2+O(\gamma^3)\Bigg]. \quad (43)$$

## 5. Related electrostatic problem

The result (31) can be used to evaluate the total charge required to raise the two caps to unit potentials in a free space. The integral equations which embody the solution of this electrostatic problem are

$$a_1^2\int_0^{\alpha_1}\int_0^{2\pi}\frac{\sin t\,\sigma_1(t)d\varphi_1 dt}{[R]_{r_1=a_1,r=a_1}}+a_2^2\int_0^{\alpha_2}\int_0^{2\pi}\frac{\sin t\,\sigma_2(t)d\varphi_1 dt}{[R]_{r_1=a_2,r=a_1}}=1\,,\quad 0\leqq\theta\leqq\alpha_1\,, \quad (44)$$

$$a_1^2\int_0^{\alpha_1}\int_0^{2\pi}\frac{\sin t\,\sigma_1(t)d\varphi_1 dt}{[R]_{r_1=a_1,r=a_2}}+a_2^2\int_0^{\alpha_2}\int_0^{2\pi}\frac{\sin t\,\sigma_2(t)d\varphi_1 dt}{[R]_{r_1=a_2,r=a_2}}=1\,,\quad 0\leqq\theta\leqq\alpha_2\,, \quad (45)$$

where $\sigma_j$, $(j=1,2)$ are the charge densities of the two caps, when these are raised to unit potentials in a free space. Comparing equations (44) and (45) with equations (8) and (9) we find that

$$\sin t\,\sigma_j(t)=\frac{1}{4\pi}[g_j(t)]_{k=0}\,, \quad (46)$$

where $g_j(t)$, $(j=1,2)$ are the solutions of integral equations (8) and (9) for the special case of section 4.

The value of the required total charge $Q$ on the two caps is

$$Q=2\pi\left[a_1^2\int_0^{\alpha_1}\sin t\,\sigma_1(t)dt+a_2^2\int_0^{\alpha_2}\sin t\,\sigma_2(t)dt\right]$$
$$=\tfrac{1}{2}\left[a_1^2\int_0^{\alpha_1}\{g_1(t)\}_{k=0}dt+a_2^2\int_0^{\alpha_2}\{g_2(t)\}_{k=0}dt\right]. \quad (47)$$

We obtain from relations (29) and (47)

$$Q=-[A(\theta)]_{k=0}\,. \quad (48)$$

Substituting in it the value of $A(\theta)$ given by (40) we obtain

$$Q=\frac{a_2}{\pi}\Bigg[(\alpha_2+\sin\alpha_2)+(\alpha_1+\sin\alpha_1)\left\{1-\frac{1}{\pi}(\alpha_2+\sin\alpha_2)\right\}^2\varepsilon+$$
$$+\Bigg\{-\frac{2}{\pi}\left(\sin\alpha_1+\frac{\sin 2\alpha_1}{2}\right)\left(\sin\alpha_2+\frac{\sin 2\alpha_2}{2}\right)\left(1-\frac{1}{\pi}(\alpha_2+\sin\alpha_2)\right)+$$
$$+\frac{1}{\pi^2}(\alpha_2+\sin\alpha_2)(\alpha_1+\sin\alpha_1)^2\left(1-\frac{1}{\pi}(\alpha_2+\sin\alpha_2)\right)^2\Bigg\}\varepsilon^2+O(\varepsilon^3)\Bigg], \quad (49)$$

which agrees with our result [12]. When we let $\alpha_1 \to \pi$ in (49), we obtain the value of $Q$ for the corresponding electrostatic problem of a sphere bounded by a concentric spherical cap in a free space.

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